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The Coulomb Branch in Gauged Linear Sigma Models

Ilarion V. Melnikov and M. Ronen Plesser

Center for Geometry and Theoretical Physics
Box 90318
Duke University
Durham, NC 27708-0318

Abstract

We investigate toric GLSMs as models for tachyon condensation in type II strings on space-time non-supersymmetric orbifold singularities. The A-model correlators in these theories satisfy a set of relations related to the topology of the resolved orbifold. Using these relations we compute the correlators and find a non-trivial chiral ring in the IR, which we interpret as supported on isolated Coulomb vacua of the theory.

1 Introduction

The study of localized closed string tachyons, first undertaken by Adams, Polchinski and Silverstein in [1], has by now a venerable history. The basic picture found by these authors and others [2–6] is that the orbifold singularity is (at least partially) resolved by tachyon condensation. A nice recent review of this and related matters is given in [7]. However, as pointed out by Martinec and Moore [8], and explored further by Moore and Parnachev [9], this cannot be the entire story. As these authors show, if one considers the D-brane charges in the orbifold theory as labeled by equivariant K-theory of the orbifold, then, naively, one might conclude that these charges *disappear* as tachyon condensation drives the system to flat space. These missing charges can be associated to branes on the isolated Coulomb vacua that are found in the Gauged Linear Sigma Model (GLSM) description of this tachyon condensation.

We follow up on this finding by studying the closed string sector of the GLSM, and we find that the missing D-brane charges and their restoration by the Coulomb branch have analogues in the closed string topological observables. We compute the correlators of the topological A-twisted model of the GLSM, and find that the isolated Coulomb vacua responsible for carrying the missing K-theory charges also support a non-trivial chiral ring. To make our explicit calculations possible, we restrict attention to a class of tractable models, as described below. We will find a rather rich set of examples amenable to our approach.

We consider localized tachyons of type II strings on space-time non-supersymmetric orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_{N(a,b,c)}$. The remarkable tractability of these backgrounds, which closely parallels the case of open string tachyons associated to unstable configurations of D-branes [10], makes this an ideal laboratory for the study of closed string tachyon condensation. These theories can be explicitly constructed as orbifolds of a free CFT [11] which leads to the identification of the localized tachyon vertex operators as certain twisted sector operators. While the GSO projection removes the bulk closed string tachyon, it leaves some of these twisted-sector operators in the theory. In addition, by suitably tuning α' and g_s, string corrections can be made small for a tachyon condensate sufficiently large to display a marked departure from the initial, unstable string vacuum. Thus, we can study tachyon condensation at string tree level, and we may hope that this captures some essential features of the endpoint of tachyon condensation.

Time evolution is difficult to study even for these simple backgrounds, and calculating the tachyon potential can be quite a challenge [12]. Instead, we study the easier problem of RG flow in the space of $d = 2$ theories: since there is a natural relevant perturbation of the orbifold CFT associated to the localized tachyon, and the IR fixed point of the flow resulting from this perturbation will be a solution of the (tree-level) string equations of motion, one might hope that the IR fixed point correctly describes the endpoint of tachyon condensation. Although it has been argued that RG flow correctly interpolates between different string backgrounds [7], and this has been shown to hold in several examples, this statement may be too strong in general. Still, it is certainly true that RG flow provides an accessible and interesting interpolation between string vacua. We will adopt this perspective and study the IR fixed points of RG flows where the UV fixed point is the orbifold CFT and

the relevant perturbation corresponds to condensing the tachyon.

The orbifold CFT is actually a superconformal theory, invariant under an $\mathcal{N} = (2,2)$ SUSY algebra. The relevant deformation corresponding to a generic tachyon completely breaks this supersymmetry, and thus, the powerful constraints of $\mathcal{N} = (2,2)$ SUSY cannot be used to constrain the RG flow. However, the deformations corresponding to tachyons that are chiral primary operators do preserve $\mathcal{N} = (2,2)$ supersymmetry. Furthermore, the most relevant tachyonic operator, i.e. the one with the largest negative mass-squared, and thus corresponding to the fastest growing instability, is a chiral primary operator [4,13]. The RG flows associated to these chiral primary tachyons are much more tractable, and in this paper we will work with these flows. We may think of this as either fine-tuning away the other tachyons, or just simplifying the analysis by studying the most unstable mode, which, after all, should grow exponentially faster than the others, and thus be the dominant effect for a substantial (RG) time.

Although we know the orbifold SCFT and the relevant perturbation of interest, the RG flow is still too difficult to study directly, mainly because it involves perturbing the theory by a twisted-sector operator. However, since these orbifolds are toric, each can be realized as the low energy limit of a simpler massive theory: a GLSM with zero superpotential. By a suitable choice of scales and parameters, we can ensure that, barring the emergence of a non-trivial separatrix, the RG flow of the GLSM passes arbitrarily near to the orbifold SCFT fixed point for many decades of the RG scale, and it then follows the relevant perturbation associated to tachyon condensation.

Our tool for exploring these models will be a study of the correlators in their topological twisted version. These encode properties of the GLSM that are interesting and yet computable. Following the seminal work of Witten [14], Morrison and Plesser provided a set of toric methods that allow us to compute certain RG invariants (*i.e.* topological observables) in these GLSMs [15]. As we will show, several surprises emerge upon taking a closer look at GLSMs that correspond to space-time non-supersymmetric backgrounds. The most impressive of these is that the instanton computation of [15] describes these correlators only in some of the phases. In others, the correlators cannot be computed by expanding in the instantons about a Higgs vacuum. These phases are additionally characterized by the existence of isolated Coulomb vacua far from the Higgs branch, and we conjecture that these support the missing correlators. Although we have not been able to confirm this by an explicit computation, we are able to show that the form of the correlators strongly suggests a Coulomb branch interpretation.

What this finding tells us about the physics of the decay process is not clear. It is difficult to understand what a set of isolated vacua of the world-sheet theory could mean in space-time. It may be relevant in this context that the models in question all involve non-compact "compactification" spaces. The picture of tachyon decay advocated in [1] describes an expanding bubble, in the interior of which the local structure is that of the resolved space, joined to the outside by some complicated structure on the bubble wall. It seems natural to guess that the Coulomb branch represents, at least at the level of the topological model, the structures that "disappear" to infinity in the RG flow. In the expanding bubble picture,

these structures would live on the bubble wall (or outside it).

The rest of the paper is organized as follows: in section 2 we briefly recall some salient features of the GLSM relevant for tachyon condensation; in section 3 we study a simple two-parameter example of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$, which encapsulates much of the essential physics of our results; in section 4 we discuss the quantum cohomology relations and the constraints they place on the topological correlators; in section 5, we use these constraints to generalize the analysis of the example, and we wrap up with a discussion in section 6.

2 The structure of the Gauged Linear Sigma Model

The power of the GLSM approach to $\mathbb{C}^3/\mathbb{Z}_{N(a,b,c)}$ theories takes root in the $\mathcal{N}=(2,2)$ world-sheet supersymmetry and its close relation to toric geometry. In this section we will outline some of the models' basic features from those perspectives.

2.1 $\mathcal{N}=(2,2)$ SUSY and the GLSM Lagrangian

We will present the field content and Lagrangian of the GLSM in $d=2$, $\mathcal{N}=(2,2)$ superspace. Labeling the four anticommuting coordinates by $\theta^\pm$,$\overline{\theta}^\pm$, the GLSM involves the following fields. The n chiral matter fields Φ^i, $i=1,\ldots,n$ have a superspace Taylor expansion

$$\Phi^i = \phi^i + \sqrt{2}\left(\theta^-\psi^i_- + \theta^+\psi^i_+\right) - 2\theta^-\theta^+F^i + \ldots, \tag{1}$$

with ϕ^i a complex scalar, $\psi^i_\pm$ right/left-moving Weyl fermions, and F^i an auxiliary complex field needed for off-shell closure of the SUSY algebra. The higher components are determined by the constraints in terms of the presented components. These matter fields are minimally coupled to u abelian gauge fields with charges Q^a_i, $a=1,\ldots,u$. The gauge fields $v_{a,\mu}$ reside in vector multiplets V_a which (in Wess-Zumino gauge) have the form

$$\begin{aligned} V_a &= \theta^+\overline{\theta}^+ v_{a,+} - \theta^-\overline{\theta}^- v_{a,-} + \Big[\sqrt{2}\theta^-\overline{\theta}^+\sigma_a \\ &\quad +2i\theta^-\theta^+\left(\overline{\theta}^+\overline{\lambda}_{a,+} + \overline{\theta}^-\overline{\lambda}_{a,-}\right) - \theta^-\theta^+\overline{\theta}^-\overline{\theta}^+ D_a + \text{c.c.}\Big], \end{aligned} \tag{2}$$

with σ_a a complex scalar, $\lambda_{a,\pm}$ left/right Weyl fermions, and D_a an auxiliary real field. The gauge field strength resides in the third and last type of multiplet that we will need: the *twisted* chiral multiplet Σ with the superspace expansion

$$\Sigma = \sigma + i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}^-\lambda_- + \sqrt{2}\theta^+\overline{\theta}^-\left(D - if_{01}\right) + \ldots, \tag{3}$$

where $f_{a,01} = \partial_0 v_{a,1} - \partial_1 v_{a,0}$ is the field strength, and σ_a, $\lambda_{a,\pm}$, and D_a are as above. (Super)gauge transformations leave Σ_a invariant, while Φ^i and V_a transform according to

$$\begin{aligned} \Phi^i &\rightarrow \exp\left(\sum_a Q^a_i\Lambda_a\right)\Phi^i, \\ V_a &\rightarrow V_a - (\Lambda_a + \overline{\Lambda}_a)/2, \end{aligned} \tag{4}$$

where Λ_a is a chiral superfield.

We define the GLSM at a scale μ by a Lagrange density $\mathscr{L}^\mu$ given by a sum of three terms, the Kähler term $\mathscr{L}^\mu_K$, the superpotential $\mathscr{L}^\mu_W$, and the *twisted superpotential* $\mathscr{L}^\mu_{\widetilde{W}}$. We take the Kähler term to be

$$\mathscr{L}^\mu_K = \int d^4\theta \left(-\frac{1}{4} \sum_{i=1}^{n} \overline{\Phi}^i \exp\left(2 \sum_{a=1}^{u} Q^a_i V_a \right) \Phi^i + \frac{1}{4\mu^2 g(\mu)^2} \sum_{a=1}^{u} \overline{\Sigma}_a \Sigma_a \right) , \tag{5}$$

where $g(\mu)$ is the dimensionless coupling of the gauge theory. The models of interest to us will have the superpotential term $\mathscr{L}_W$ set to zero, and for reasons that will be made clear below, we will call such theories *toric* GLSMs. Finally, the tree-level twisted superpotential is given by

$$\mathscr{L}^\mu_{\widetilde{W}} = \left[-\frac{i}{2\sqrt{2}} \int d\theta^+ d\overline{\theta}^- \sum_{a=1}^{u} \Sigma_a \tau^a(\mu) \right] + \text{c.c.}. \tag{6}$$

The $\tau^a(\mu) = ir^a(\mu) + \frac{\theta^a}{2\pi}$ are the parameters of the model. Each τ^a is a combination of the Fayet-Iliopoulos (F-I) term r^a and the θ-angle θ^a. It is useful to define single-valued parameters $q_a = e^{2\pi i \tau^a}$.

$\mathscr{L}^\mu$ provides a good description of the degrees of freedom and their interactions at external momenta of order μ. The GLSM is asymptotically free, and the low energy theory is strongly coupled in terms of these degrees of freedom. For more details about the structure of $\mathcal{N} = (2,2)$ SUSY and the GLSM Lagrangian the reader should consult [14, 16].

2.2 The phases of the GLSM, I

Although quantum effects substantially modify the low energy description of the theory, it pays to consider the classical moduli space of the GLSM. The classical moduli space is the set of supersymmetric vacua, parametrized, as usual, by the zeroes of the classical scalar potential modulo the gauge group. The toric GLSM has the scalar potential

$$U(\phi^i, \sigma_a) = 2 \sum_{i,a,b} |\phi^i|^2 Q^a_i Q^b_i \sigma_a \overline{\sigma}_b + \frac{1}{2\mu^2 g(\mu)^2} \sum_a (D^a)^2 , \tag{7}$$

with

$$D^a = \mu^2 g(\mu)^2 \left(\sum_i Q^a_i |\phi^i|^2 - r^a \right) . \tag{8}$$

The classical analysis is straightforward. Despite the lack of a dynamical two-dimensional photon or spontaneous symmetry breaking in $d = 2$, we can describe the classical physics as the Higgs mechanism.[1] Generically, the gauge group is completely Higgsed, σ_a are all massive, and solving for $D^a = 0$ for all a modulo the gauge group, one finds that the

[1]Going beyond the classical theory, one could imagine this as the starting point for a semi-classical Born-Oppenheimer analysis of the vacuum structure.

moduli space is a complex dimension $d = n - u$ toric variety, whose geometric properties are independent of the θ-angles and vary smoothly with the u F-I parameters r^a. The geometry degenerates as certain cycles shrink to zero size along co-dimension one walls in the $\mathbb{R}^u$ space spanned by the r^a. This process is a physical realization of the familiar blow-ups and blow-downs of algebraic (for our purposes toric) geometry. The regions of the parameter space separated by the walls are termed the *phases* of the GLSM. The phases turn out to be the full-dimensional cones in the secondary fan associated to the toric fan of the GLSM.

When the F-I terms $r^a(\mu)$ are deep in the interior of a particular phase $\mathcal{K}$, corresponding to a classical moduli space V, then the light fields correspond to those of the NLSM with a target space that is topologically equivalent to V. Furthermore, these light fields are weakly coupled to the massive fields. Thus, by choosing $r^a(\mu)$ sufficiently deep in $\mathcal{K}$, we can ensure that the RG trajectory of the GLSM passes arbitrarily close to that of a NLSM with target space V. In particular, by appropriately choosing a GLSM and $r^a(\mu)$ we can ensure that the high energy theory is well approximated by an orbifold theory of the form $\mathbb{C}^3/\mathbb{Z}_{N(a,b,c)}$. As explained in [8], the necessary limit is to choose a fiducial scale μ^* and send $g(\mu^*) \to \infty$, while holding $r^a(\mu^*)$ fixed (and in the correct phase).

On the other hand, when $r^a(\mu)$ approaches a phase boundary, the description as a NLSM with target space V seems to break down. As we discuss in the next section, this is not entirely as it seems.

2.3 Quantum moduli space and the one loop β-function

Quantum effects modify the classical phases picture in several important ways. The most basic is the one loop renormalization of the F-I parameters:

$$\mu \frac{\partial}{\partial \mu} r^a = \frac{1}{2\pi} \sum_i Q_i^a. \tag{9}$$

World-sheet supersymmetry ensures that this result is uncorrected in perturbation theory. More precisely, holomorphy ensures that τ^a is uncorrected beyond the shift above.

It is easy to show that by a $\mathrm{GL}(u, \mathbb{Z})$ transformation one can always choose a basis for the Q_i^a such that $\sum_i Q_i^1 = \Delta$, and $\sum_i Q_i^a = 0$ for $a = 2, \ldots, u$. Such a change of basis leaves the classical moduli space unchanged, and it preserves the IR physics, since it only changes the structure of the gauge field kinetic terms—an irrelevant Kähler deformation. Let us assume that such a basis is chosen. If $\Delta = 0$, then r^1 does not run, and each phase is bi-rationally equivalent to a toric Calabi-Yau manifold. The resulting string theory "compactification" is supersymmetric and fairly well understood [15, 17]. We will be interested in models where $\Delta \neq 0$. In that case, at low energies the model is driven to $r^1 = -\infty$ if $\Delta > 0$ and to $r^1 = +\infty$ if $\Delta < 0$.[2] As discussed in [2, 4, 8], this is the RG flow which corresponds to

[2]Obviously, r^1 runs and should not really be considered a parameter of the theory. Similarly, the corresponding θ-angle θ^1 is not a parameter either: the RG running is accompanied by an anomaly in the axial R-symmetry, which breaks the $U(1)_A$ to $\mathbb{Z}_{|\Delta|}$. This anomaly allows us to to set θ^1 to a Δ-th root of unity.

tachyon condensation in space-time. It is convenient to express this running in terms of q_1:

$$q_1(\mu) = q_1(\mu_0)\left(\frac{\mu_0}{\mu}\right)^{\Delta}. \tag{10}$$

While q_1 runs, the q_a for $a > 1$ are RG invariants and parametrize exactly marginal deformations of the GLSM.

In addition, quantum effects change the nature of the classical singularities [14,15]. Recall that these singularities in the low energy NLSM description were due to the appearance of massless σ^a along co-dimension 1 walls in the space of r^a. Each wall is associated to the un-Higgsing of a particular gauge group. Suppose the charges of the un-Higgsed gauge group are Q_i, the complex scalar is σ, and the complex parameter is q. Furthermore, suppose that q is tuned to the vicinity of the classical singularity, $|q| = 1$. When $\sum_i Q_i \neq 0$, quantum effects generate a potential for σ, and thus smooth out the classical singularity. On the other hand, if $\sum_i Q_i = 0$, then for a particular value of q, there is no potential generated for σ, and thus a continuous Coulomb branch emerges. This Coulomb branch is parametrized by the expectation value of σ. The semi-classical analysis leading to this result is valid when $|\sigma|$ is large and the other gauge groups are Higgsed with large F-I terms. This analysis shows that the *singular locus* of the theory, i.e. the values of the q^a where the low energy NLSM description breaks down, is a *complex* co-dimension one subvariety in the space of the GLSM parameters $q^a \in \mathbb{C}^u$. Thus, all the phases are connected by paths along which the low energy NLSM description is non-singular.

2.4 The Quantum Coulomb branch

In addition to the continuous Coulomb branch which exists on the singular locus, when $\Delta \neq 0$, the GLSM possesses another set of Coulomb vacua. These are isolated vacua, whose locations vary continuously with the GLSM parameters. To study the Coulomb branches in more detail, it pays to return to the classical scalar potential of eqn. (7). The form of the potential indicates that when the σ_a acquire non-zero expectation values, they give masses to some or all of the ϕ^i. We may integrate out these massive chiral matter fields to obtain an effective action for the σ_a appropriate at sufficiently low energies. In the case that all the matter fields are massive, $\mathcal{N} = (2,2)$ supersymmetry combined with 't Hooft anomaly matching determine the effective twisted superpotential at scale μ to be [15]

$$\widetilde{W}_{\text{eff}} = -\frac{1}{4\pi\sqrt{2}}\sum_{a=1}^{u}\Sigma_a \log\left[\prod_{i=1}^{n}\left(\frac{1}{\exp(1)\mu}\sum_{b=1}^{u}Q_i^b\Sigma_b\right)^{Q_i^a}/q_a\right]. \tag{11}$$

When is this effective description valid? It is not valid at high energies when $r^a(\mu)$ are deep in the interior of a phase. Here we know that there are light ϕ degrees of freedom that are certainly missed in the $\widetilde{W}_{\text{eff}}$ description. However, it is perfectly plausible for a Coulomb branch to emerge at low energies or at the singularities on the phase boundaries. Indeed,

the former is necessary to match the UV and IR computations of the Witten index, and the latter provides an explanation of how the GLSM resolves the NLSM singularities.

The equations of motion that follow from $\frac{\partial \widetilde{W}_{\rm eff}}{\partial \sigma^a} = 0$ are

$$\prod_i \left(\frac{1}{\mu} \sum_b Q_i^b \sigma_b \right)^{Q_i^a} = q_a. \tag{12}$$

We may write these as polynomial relations among the σ_a:

$$\prod_{i|Q_i^a>0} \left(\sum_b Q_i^b \sigma_b \right)^{Q_i^a} - \mu^{\Delta_a} q_a \prod_{i|Q_i^a<0} \left(\sum_b Q_i^b \sigma_b \right)^{-Q_i^a} = 0, \tag{13}$$

where, as before, $\Delta_a = \sum_i Q_i^a$. To be precise, the last two equations are equivalent only if the two products in the second equation have no common factors. This will be an assumption in what follows, but we will see that it is satisfied in a wide class of examples.

For some purposes it is convenient to choose a standard basis of Q_i^a such that $\sum_i Q_i^1 = \Delta$ and $\Delta_a = 0$ for $a > 1$. Making this choice, and letting

$$\begin{aligned} \omega_a &= \sigma_a/\sigma_1 \quad , a > 1, \\ \zeta_i &= Q_i^1 + \sum_{a>1} Q_i^a \omega_a, \\ s(\omega_2, \ldots, \omega_u) &= \prod_i \zeta_i^{-Q_i^1}, \end{aligned} \tag{14}$$

the equations can be written as

$$\begin{aligned} P_a(\omega_2, \ldots, \omega_u) &= \prod_{Q_i^a>0} \zeta_i^{Q_i^a} - q_a \prod_{Q_i^a<0} \zeta_i^{-Q_i^a} = 0 \quad a > 1, \\ \sigma_1^\Delta &= \mu^\Delta q_1 s(\omega_2, \ldots, \omega_u), \\ \sigma_a &= \omega_a \sigma_1, \quad a > 1. \end{aligned} \tag{15}$$

The case $u = 2$ is simple, and since we will study it in detail below, we will specialize the above expressions to it. There is a single ω, which takes on the $\deg(P)$ values of the roots of a single polynomial $P(\omega)$. The degree of $P(\omega)$ is given by $\deg(P) = \sum_{Q_i^2>0} Q_i^2$. For $u = 2$ and generic q_2 there are $|\Delta| \deg(P)$ distinct σ-vacua. It is no accident that the σ-vacua come in families of $|\Delta|$ vacua related by roots of unity: the axial $U(1)$ R-symmetry, which is anomalous in the high energy description of the GLSM, is spontaneously broken in these Coulomb vacua to $\mathbb{Z}_{|\Delta|}$. Note that the condition that eqn. (12) is equivalent to eqn. (13) is that $P(\omega)$ is irreducible over $\mathbb{Z}$.

We have seen how the twisted superpotential produces the isolated Coulomb vacua. Of course, it may also be used to study the emergence of the continuous Coulomb branch. The $\widetilde{W}_{\rm eff}$ given above was derived under the assumption that all of the matter fields are massive.

This need not be the only possibility. It is possible to find vacua where some of the ϕ^i are massive, while others have expectation values that, in turn, give masses to some of the σ_a. In particular, working in the basis of Q_i^a given above, one can see that there may be vacua where the gauge group corresponding to Q_i^1 is Higgsed, while the rest of the σ_a are massless. The resulting effective potential for the σ_a, $a > 1$ will be a set of $u-1$ homogeneous equations with $u-1$ parameters, leading to an implicit form for a component of the singular locus. When q_a are tuned to this variety, a continuous Coulomb branch will arise, with $\sigma_1 = 0$, and σ_a large for $a > 1$.

It is important not to confuse the different types of Coulomb vacua, so we will end by re-iterating some of the differences between the two. The isolated Coulomb vacua arise only when $\Delta \neq 0$. They are found in the IR phases of the GLSM, where they provide a set of supersymmetric vacua that vary smoothly with the parameters q_a. On the other hand, continuous Coulomb vacua only arise in GLSMs where it is possible to un-Higgs a gauge group with charges satisfying $\sum_i Q_i = 0$. They occur only on the singular locus of the model.

In this paper, we will be largely concerned with the isolated Coulomb vacua, so unless indicated otherwise, "Coulomb branch" will mean the set of these vacua.

2.5 The phases of the GLSM, II

In summary, we find the following picture for the GLSM phases. By suitably tuning $g(\mu^*)$ and $r^a(\mu^*)$ at some fiducial scale μ^*, we ensure that the GLSM is in a phase where its low energy behavior is well approximated by a NLSM with a non-compact target space V for many decades of the RG scale μ.[3] The GLSM is well described by a purely "Higgs branch" description, again in the sense that for many decades of μ the physics is insensitive to the eventual emergence of a Coulomb branch and a new Higgs branch.

At low energies, the high energy degrees of freedom of the GLSM become strongly coupled, and a different effective description is needed. Despite the asymptotic freedom of the GLSM, the IR fixed point may, to some extent, be described by the GLSM degrees of freedom. On the one hand, the IR Higgs branch is again weakly coupled, since after passing through a strongly coupled region ($r^a(\mu) \approx 0$), the $r^a(\mu)$ are again deep in the interior of another phase, so the statements about light fields corresponding to a NLSM with target space V' still hold. On the other hand, the Coulomb branch can be described by the effective twisted superpotential, whose form is completely fixed by 't Hooft anomaly matching and supersymmetry.

A final point we would like to highlight is the question of the separation between the isolated Coulomb and Higgs vacua. Essentially, we are in the usual trouble: a SUSY theory is quite revealing of its zero energy states, but it keeps non-zero energy information mostly to itself, and questions about energy barriers between various vacua remain difficult. Naively, one may argue that the D-terms of the Higgs branch are large if evaluated on the isolated Coulomb branch. Or, one may feel that a better argument is that the two sets of vacua are

[3] The case of most interest to us will be $V \simeq \mathbb{C}^3/\mathbb{Z}_{N(a,b,c)}$.

well separated in field space.[4] However, both arguments are affected by the renormalization of the Kähler terms in the theory. Does this renormalization qualitatively change the physics? While it is tempting to say that only the development of a singularity could cause the branches to come arbitrarily close in field space and/or decrease the energy barriers between them to be arbitrarily small, we cannot make a rigorous statement about this.

2.6 The topological A-twist and instanton sums

Our understanding of quantum effects is not so complete that we can directly compute any quantities along the RG flow. Our strategy is to follow an old idea of Witten: working on a world-sheet with Euclidean signature, we can use a related topological (and hence RG-invariant) theory to compute a set of RG-invariants in the original theory. These RG invariants can then be used to probe non-perturbative aspects of the IR physics. The background for this approach is well covered in [14, 15, 18], and we will content ourselves with a cursory treatment of it here.

The existence of this topological theory, the so-called A-model, follows from the structure of the $U(1)_R \times U(1)_L$ R-symmetry subgroup of the $\mathcal{N} = (2,2)$ supersymmetry. The R-symmetry leaves the Φ^i and V_a superfields invariant, but rotates the $\theta^{\pm}$ with charges $Q_R(\theta^+) = Q_L(\theta^-) = 1$, $Q_R(\theta^-) = Q_L(\theta^+) = 0$. All other charges are determined by this choice, and, in particular, we see that

$$Q_R(\sigma_a) = -Q_L(\sigma_a) = 1. \tag{16}$$

The axial $U(1)_A$ subgroup with charges $Q_A = \frac{1}{2}(Q_R - Q_L)$ is anomalous, with

$$\Delta Q_A = \sum_{i,a} n_a Q_i^a, \tag{17}$$

where

$$n_a = -\frac{1}{2\pi}\int_\Sigma f_a \tag{18}$$

is the instanton number associated to the a-th gauge group on the world-sheet $\Sigma \simeq \mathbb{P}^1$.[5] The vector $U(1)_V$ is non-anomalous and can be used to twist the theory along the lines of [14]. We modify the spin connection J_T by defining a new Lorentz current $J_T' = J_T + J_V$. This is the A-twist. The B-twist, associated to twisting by the anomalous axial symmetry does not give rise to a well-defined theory when $\Delta \neq 0$.

The twisted theory has several important properties, which we now review. First, twisting renders a linear combination of the $\mathcal{N} = (2,2)$ an anticommuting (world-sheet) scalar operator $\mathcal{Q}$, which can be used as a BRST-like operator to project the observables of the theory

[4] Note that while using classical D-terms prompt one to argue that the isolated Coulomb vacua are separated from the Higgs vacua, it would also suggest that the continuous Coulomb branch may be connected to the Higgs branch.

[5] Note that Q_i^a and r^a are naturally thought of as vectors in $\mathbb{R}^u$, and n_a is a vector in the dual space $(\mathbb{R}^u)^\vee$.

onto its cohomology—the topological observables. These topological observables decouple from $\mathcal{Q}$-exact operators in correlation functions and are nothing other than the twisted versions of chiral operators of the untwisted theory. In the GLSM, the set of local observables in the $\mathcal{Q}$-cohomology is spanned by powers of σ_a. The $U(1)_A$ remains an anomalous symmetry of the theory, with ΔQ_A modified to be

$$\Delta Q_A = d + \sum_{i,a} n_a Q_i^a, \tag{19}$$

where, as before, $d = n - u$. Q_A is called *the ghost number* in analogy with the usual BRST ghost number.

Second, one can show that a perturbation of the world-sheet metric corresponds to modifying the action by a $\mathcal{Q}$-exact term, so that correlation functions of topological observables are insensitive to the world-sheet metric, and hence, are truly topological. It immediately follows that the correlation functions are independent of the positions of the operator insertions on the world-sheet.

In addition, a perturbation of the coupling $g(\mu)$ corresponds to a $\mathcal{Q}$-exact change in the action, indicating that the correlators may be computed at weak coupling, where the action localizes onto $\mathcal{Q}$-invariant field configurations, which in the case of the toric GLSM are given by the field configurations satisfying

$$\begin{aligned} d\sigma_a &= 0, \\ \sum_a Q_i^a \sigma_a \phi^i &= 0, \\ D_a + f_a &= 0, \\ D_{\bar{z}} \phi^i &= 0 \end{aligned} \tag{20}$$

modulo the action of the gauge group. Morrison and Plesser have argued that this space of field configurations is a union of disconnected moduli spaces $\mathcal{M}_{\mathbf{n}}$ labeled by the u instanton numbers n_a. Each $\mathcal{M}_{\mathbf{n}}$ is a toric variety which can be explicitly constructed as a holomorphic quotient. Furthermore, they showed that, fixing a phase with cone $\mathcal{K}$, the correlation functions

$$Y_{a_1 \cdots a_s} = \langle \sigma_{a_1}(z_1) \cdots \sigma_{a_s}(z_s) \rangle \tag{21}$$

may be computed as a sum over $\mathbf{n}$ in the dual cone $\mathcal{K}^\vee$:

$$Y_{a_1 \cdots a_s} = \sum_{\mathbf{n} \in \mathcal{K}^\vee} Y^{\mathbf{n}}_{a_1 \cdots a_s} \prod_a q_a^{n_a}, \tag{22}$$

where $Y^{\mathbf{n}}_{a_1 \cdots a_s}$ is a coefficient independent of $\mathbf{q}$, and the factor $\prod_a q_a^{n_a}$ corresponds to $\exp(-S)$ evaluated on a field configuration belonging to $\mathcal{M}_{\mathbf{n}}$. Recall that $\mathcal{K}^\vee$ is defined by

$$\mathcal{K}^\vee = \left\{ \mathbf{n} \in (\mathbb{R}^u)^\vee \,|\langle \mathbf{n}, \mathbf{r} \rangle \geq 0 \ \text{ for all } \ \mathbf{r} \in \mathcal{K} \right\}. \tag{23}$$

From the form of $\mathcal{K}^\vee$ it follows that the sum in eqn. (22) converges for $\mathbf{q}$ corresponding to $\mathbf{r}$ in the interior of $\mathcal{K}$, with the radius of convergence determined by the distance to the

closest singularity. The coefficients $Y^{\mathbf{n}}_{a_1\cdots a_s}$ are determined as intersection numbers on the toric variety $\mathcal{M}_{\mathbf{n}}$. These correlators are subject to conservation of ghost number, eqn. (19), which implies

$$Y^{\mathbf{n}}_{a_1\cdots a_s} = 0 \quad \text{for} \quad s \neq d + \sum_{a,i} n_a Q_i^a. \tag{24}$$

We will not review the details of the computation of the non-zero $Y^{\mathbf{n}}_{a_1\cdots a_s}$ in this section, but we will highlight aspects of the computation that differ somewhat from the results of Morrison and Plesser. Briefly, the method of [15] is to recast the Kähler quotient of eqn. (20) as a holomorphic quotient:

$$\mathcal{M}_{\mathbf{n}} = \frac{\bigoplus_i H^0(\mathscr{O}(d_i)) - F_{\mathbf{n}}}{(\mathbb{C}^*)^u}, \tag{25}$$

where $d_i = \langle \mathbf{n}, \mathbf{Q}_i \rangle = \sum_a n_a Q_i^a$, $\mathscr{O}(n)$ is a holomorphic line bundle of degree n over $\mathbb{P}^1$, $H^0(\mathscr{O})$ is the space of holomorphic sections of $\mathscr{O}$, and $F_{\mathbf{n}}$ is a certain excluded set, presented as an irreducible union of intersections of hyperplanes.

The $Y^{\mathbf{n}}_{a_1\cdots a_s}$ are given by

$$Y^{\mathbf{n}}_{a_1\cdots a_s} = f(\mu)\frac{1}{|H|}\langle \hat{\sigma}^{(\mathbf{n})}_{a_1} \cdots \hat{\sigma}^{(\mathbf{n})}_{a_s} \chi_{\mathbf{n}} \rangle_{\mathcal{M}_{\mathbf{n}}}, \tag{26}$$

where $f(\mu)$ is a factor associated to twisting the theory, H is (the possibly trivial) discrete subgroup of the gauge group left unbroken in the phase $\mathcal{K}$, the $\hat{\sigma}^{(\mathbf{n})}_a$ are proportional to toric divisors $\sigma^{(\mathbf{n})}_a$ on $\mathcal{M}_{\mathbf{n}}$, and $\chi_{\mathbf{n}}$ is the Euler class of a certain obstruction bundle, whose appearance is described in [15, 19]. The intersection computations can be performed with standard toric methods.[6]

The form of $f(\mu)$ and the relation between $\sigma^{(\mathbf{n})}$ and $\hat{\sigma}^{(\mathbf{n})}$ could be fixed by carrying out the path integral localization explicitly. However, to date, this has not been done. Nevertheless, we can determine the correct scaling by appealing to the simple fact that the topological correlators are RG invariants, and thus should not depend on μ or $q_1(\mu)$ separately, but only through the RG-invariant combination $\mu^{\Delta} q_1(\mu)$. A natural guess that satisfies this criterion is $\hat{\sigma}^{(\mathbf{n})} = \mu\sigma^{(\mathbf{n})}$ and $f(\mu) = \mu^{-d}$. This scaling is a difference from [15] that we would like to emphasize. In the case of [15], where the primary interest concerned computations in Calabi-Yau (i.e. superconformal) GLSMs, this scaling did not play an important role, since the ghost number selection rule restricted non-zero correlators to $s = d$, and q_1 was a parameter of the model. It would be nice to determine $f(\mu)$ and the scaling of $\sigma^{(\mathbf{n})}$ by an explicit computation.

Another important point is that $\mathcal{M}_{\mathbf{n}=0} \simeq V$, the toric variety corresponding to $\mathcal{K}$. We will be interested in non-compact V, for which the intersection computations are not well-defined, and we will need to rely on another method to determine the zero instanton contributions.

The most important difference from the superconformal case concerns the validity of the instanton expansion in various phases. While in the superconformal case *any* phase could be

[6] More details on the geometry of $\mathcal{M}_{\mathbf{n}}$ and the form of $\chi_{\mathbf{n}}$ may be found in Appendix A.

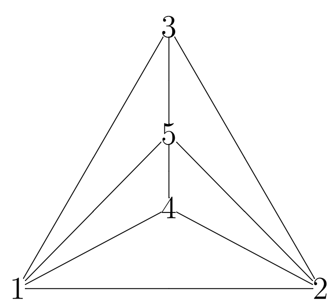
used to compute the topological correlators via the instanton sum, in general, the instanton computation is only appropriate in certain phases. When we compute in the high energy phase of the GLSM, where the Higgs branch is a good description of the semi-classical vacua, the instanton expansion is valid. However, as we learned above, the low energy phases also possess a Coulomb branch, and an instanton expansion in that phase will inevitably miss these vacua.

3 An example: $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$

Having reviewed the basic physics of the toric GLSM and its A-model, we will now apply these ideas to a concrete example. We build the GLSM corresponding to $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$ by starting with a minimal resolution of this toric singularity. The toric construction of these orbifolds is well explained in [20]. The toric fan of this minimal resolution has its one-dimensional rays labeled by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \\ 2 & 2 & 1 \\ 1 & 1 & N+1 \end{pmatrix}, \tag{27}$$

where $M = 2N+1$. The combinatorics of the fan are as follows:



The GLSM gauge charges are a basis for the relations among the v_i. A convenient choice is

$$Q = \begin{pmatrix} 1 & 1 & 1 & -N & -1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}. \tag{28}$$

There are four classical phases, each corresponding to a triangulation of the fan. The fan without any subdivisions is the orbifold phase, the partially subdivided fans correspond to partial resolutions, and the completely subdivided fan is the smooth phase. These phases are depicted in Fig. (1). The RG running of this GLSM is determined by $\Delta = \sum_i Q_i^1 = 2 - N$: for $N > 2$ the model driven to small $|q_1|$ or, equivalently, large r^1. The figure encapsulates much of the the basic physics discussed above: the UV fixed point is an orbifold, and RG flow leads to a less singular geometry; when $|q_1|$ is small and $q_2 = 1/4$, semi-classical analysis shows that a continuous Coulomb branch, parametrized by the expectation value of σ_2 emerges.

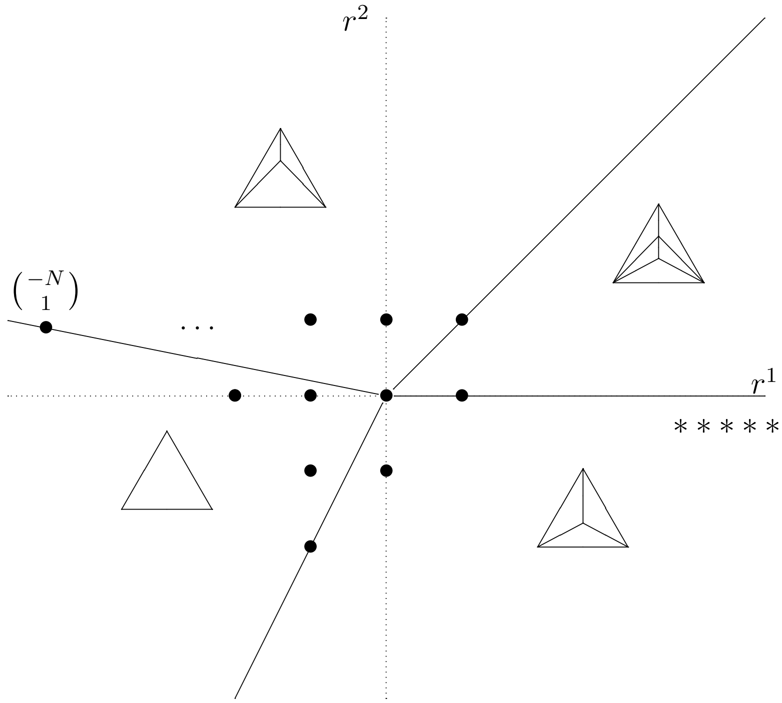


Figure 1: Phases of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$. The $*****$ indicates the projection of the semi-classical singularity to the (r^1, r^2)-plane.

3.1 Topological correlators

The topological correlators of the model are $Y_{ab} = \langle \sigma_1^a \sigma_2^b \rangle$. The ghost number anomaly implies that $Y_{a,b} = 0$ if $a + b \neq 3 + (2 - N)n_1$ for some n_1. This immediately leads to a puzzle: while expanding in the instantons in the orbifold phase, where $\mathbf{n} \in \mathcal{K}^\vee$ may have arbitrarily *negative* n_1, leads to an infinite number of non-trivial correlators, the instantons in the smooth phase, where $\mathbf{n} \in \mathcal{K}^\vee$ requires $n_1 \geq 0$, cannot support most of these. Based on the experience with Calabi-Yau GLSMs, this would be confusing, because these correlation functions are RG invariants, and should be computable in any phase. Furthermore, since no singularities separate the phases from one another, we expect to be able to analytically continue the correlation functions from any one phase to another. In fact, it is known that the $Y_{a,b}$ are rational functions in the $q_{1,2}$, so that this analytic continuation is unambiguous.

Of course, there is a natural resolution to this puzzle: the presence of isolated Coulomb vacua deep in the interior of a phase, precisely where an instanton computation is expected to be trustworthy, means that the instanton computations may not be reliable in these phases. The Coulomb vacua may not only invalidate a Higgs branch computation but also support the missing correlators.

3.1.1 The instanton sums

Using the methods of [15], we are able to explicitly compute the non-zero correlators in the orbifold phase:

$$Y_{a,b} = q_1(\mu)^{n_1} \sum_{n_2=Nn_1}^{\lfloor -\frac{n_1}{2} \rfloor} q_2^{n_2} Y_{a,b}^{n_1,n_2}, \quad \text{with} \quad a+b = 3+(2-N)n_1. \tag{29}$$

The notation $\lfloor a \rfloor$ ($\lceil a \rceil$) signifies the greatest (smallest) integer less (greater) than a and the coefficients are given by

$$\begin{aligned} Y_{a,b}^{n_1,n_2} &= \mu^{n_1(2-N)} \tfrac{1}{M} \oint_{C(0)} \frac{dz}{2\pi i} \frac{(2+z)}{z(1+(N+1)z)} \left[\frac{(-M)^{N+1} z}{(2+z)^N (1+(N+1)z)} \right]^{n_1} \times \\ &\times \left[\frac{-Mz^2}{1+(N+1)z} \right]^{n_2} \left[\frac{-1+Nz}{2+z} \right]^{b}, \end{aligned} \tag{30}$$

where, as before, $M = 2N+1$ and$C(0)$ is a contour about $z = 0$ that avoids all other poles of the integrand.[7] As mentioned above, the methods of [15] are reliable only for $n_1 \neq 0$. The finite sum on n_2 implies that these correlators can only be singular as $q_2 \to \infty$ or $q_2 \to 0$. On the other hand, the instanton sums for the $n_1 = 0$ correlators seem to have just the $\mathbf{n} = 0$ term, and hence should just be constants. So, the semi-classical singularity at $q_2 = \frac{1}{4}$ is another puzzle, since there do not seem to be correlators sensitive to it.

In order to compute the $n_1 = 0$ correlators, as well as to make sense of the puzzle of missing correlators, we can appeal to a remarkable set of identities satisfied by the correlators. Let $\mathscr{O}$ be an A-model observable of this GLSM. Then, using eqn. (30), it is easy to show that the $n_1 \neq 0$ correlators satisfy

$$\begin{aligned} \langle (\sigma_1 + \sigma_2)(\sigma_2 - N\sigma_1)\, \mathscr{O} \rangle &= q_2 \langle (\sigma_1 + 2\sigma_2)^2\, \mathscr{O} \rangle, \\ \langle \left(\sigma_1^3 + \sigma_1^2 \sigma_2 \right) \mathscr{O} \rangle &= -\mu^{(2-N)} q_1(\mu) \langle (\sigma_2 - N\sigma_1)^N (\sigma_1 + 2\sigma_2)\, \mathscr{O} \rangle. \end{aligned} \tag{31}$$

We will show below that these relations hold more generally and are quantum cohomology relations on topological correlators, whose form is determined by the effective twisted superpotential. Assuming that these relations are satisfied by the $n_1 = 0$ correlators as well as the $n_1 \neq 0$ ones, it is straightforward to use the relations to determine the $Y_{3-b,b}$ correlators:

$$\begin{aligned} Y_{3,0} &= \frac{2}{M}, \\ Y_{2,1} &= -\frac{1}{M}, \\ Y_{1,2} &= \frac{2q_2 - N - 1}{M(4q_2 - 1)}, \\ Y_{0,3} &= \frac{-4q_2^2 + (5+6N)q_2 + N^2 - N - 1}{M\,(4q_2 - 1)^2}. \end{aligned} \tag{32}$$

[7]Details of the computation may be found in appendix B.

Thus, we find that the $Y_{1,2}$ and $Y_{0,3}$ are the correlators sensitive to the $q_2 = \frac{1}{4}$ singularity. We stress that this q_2 dependence means that these correlators are not computed by the standard instanton sums.

3.2 Relations and missing correlators

The relations in eqn. (31) are powerful. We can easily show that they completely determine all the instanton correlators in terms of *two* of them, say $Y_{3,0}$ and $Y_{2,1}$. This may well be the most efficient procedure for evaluating A model correlators in toric GLSMs, and we will return to it below. However, they are also interesting for our present purpose of understanding the "disappearing" of the Higgs phase correlators.

There is a simple way to write down a set of correlators that obeys eqn. (31). It is based on the isolated Coulomb vacua of the GLSM. The equations of motion that follow from the twisted superpotential of this model are (compare to eqn. (31))

$$\begin{aligned}
(\sigma_1+\sigma_2)(\sigma_2-N\sigma_1) &= q_2(\sigma_1+2\sigma_2)^2, \\
\left(\sigma_1^3+\sigma_1^2\sigma_2\right) &= -\mu^{(2-N)}q_1(\mu)\left(\sigma_2-N\sigma_1\right)^N(\sigma_1+2\sigma_2).
\end{aligned} \tag{33}$$

As discussed in section 2.4, there are $N_v = 2(N-2)$ isolated vacua. Let us label these by $(\sigma_{1,\alpha}, \sigma_{2,\alpha})$, $\alpha = 1, \ldots, N_v$. A set of functions that obey the relations and are rational functions of the q_a is given by

$$Y^{\mathrm{Coul}}_{a,b} = \sum_{\alpha=1}^{N_v} Z(\sigma_{1,\alpha}, \sigma_{2,\alpha})\sigma^a_{1,\alpha}\sigma^b_{2,\alpha}, \tag{34}$$

with $Z(\sigma_1, \sigma_2)$ an undetermined function. Recall that the $(\sigma_{1,\alpha}, \sigma_{2,\alpha})$ can be parametrized as in eqn. (15). In this case, we find

$$\begin{aligned}
P(\omega_\pm) &= (1+\omega_\pm)(\omega_\pm-N) - q_2(1+2\omega_\pm)^2 = 0, \\
\sigma_{1,\pm;p} &= \lambda^p\mu\left(q_1(\mu)s(\omega_\pm)\right)^{\frac{1}{2-N}}, \\
\sigma_{2,\pm;p} &= \omega_\pm\sigma_{1,\pm;p},
\end{aligned} \tag{35}$$

where $\lambda = e^{\frac{2\pi i}{N-2}}$, $p = 0, \ldots, N-1$, and

$$s(\omega) = \frac{(-1-2\omega)(\omega-N)^N}{1+\omega}. \tag{36}$$

If, in addition to the relations, we demand that the selection rule $a + b = 3 + (2-N)n_1$ holds for non-zero correlators, the function Z is constrained to have the form

$$Z(\sigma_{1,\pm;p}, \sigma_{2,\pm;p}) = \sigma^{-3}_{1,\pm;p}F(\omega_\pm)/(N-2), \tag{37}$$

for some $F(\omega)$. The two instanton correlators that are left undetermined by the relations are just right to fix the form of $F(\omega_\pm)$, and the correlators can be written as in eqn. (34) with

$$Z(\sigma_1, \sigma_2) = \frac{1}{(N-2)\sigma_1^2(2\sigma_2N+(3N+1)\sigma_1)}. \tag{38}$$

The form of $Y_{a,b}$ that follows from $Y_{a,b} = Y_{a,b}^{\mathrm{Coul}}$ is suggestive of a correlator computation on the Coulomb branch. Recall, that the Coulomb vacua are massive, and with eqn. (20) precluding tunneling between the vacua ($d\sigma = 0$), it is sensible that the correlators are given simply by products of the vacuum expectation values of the fields. Furthermore, the appearance of Z is also natural: we expect that the localization of the GLSM path integral in the A-model and the integrating-out of the chiral matter and the anticommuting fields will lead to a non-trivial measure for the σ_a path integral.

This then is the resolution of the puzzle: the correlators that were computed in the orbifold phase as an instanton expansion about a Higgs vacuum do not disappear. Instead, they become correlators supported on the Coulomb vacua.

4 Quantum cohomology relations and the Coulomb branch

In this section we study the relations of eqn. (31) in a more general setting. The general form of these relations follows from eqn. (13). Letting $\xi_i = \sum_a Q_i^a \sigma_a$, we have

$$\langle \mathscr{O} \prod_{Q_i^a > 0} \xi_i^{Q_i^a} \rangle = \mu_a^{\Delta} q_a(\mu) \langle \mathscr{O} \prod_{Q_i^a < 0} \xi_i^{-Q_i^a} \rangle \quad \text{for all} \quad a, \tag{39}$$

where $\mathscr{O}$ is, an A-model observable. These *quantum cohomology relations* are proved in Appendix C. The proof assumes that there exists some phase $\mathcal{K}$ where the correlators may be computed by instanton sums. Working in this phase, we are able to reduce eqn. (39) to statements about the cohomology rings of the moduli spaces $\mathcal{M}_{\mathbf{n}}$. Given the toric structure of $\mathcal{M}_{\mathbf{n}}$, we show that these statements follow from a few simple combinatoric observations.

We can show that the requisite $\mathcal{K}$ exists in general GLSMs. The phase $\mathcal{K}$ must be one of the UV phases of the GLSM. Working with a standard basis, we see that in a UV phase $r^1 = |r^1| \operatorname{sign}(\Delta)$. The classical D-term equations for $\mathbf{r} \in \mathcal{K}$ take the form

$$\begin{aligned} \sum_i Q_i^1 |\phi^i|^2 &= |r^1| \operatorname{sign}(\Delta), \\ \sum_i Q_i^a |\phi^i|^2 &= r^a, \quad \text{for } a > 1. \end{aligned} \tag{40}$$

The first of these obviously has a solution for arbitrarily large $|r_1|$, and the others have the property that if there is no solution for r^a, then there exists a solution for $-r^a$. Thus, there exists a UV phase $\mathcal{K}$ where, by suitably tuning the F-I parameters, the classical Higgs branch can be made to be an arbitrarily good description of the physics, and thus, an instanton computation should be reliable.

As in the example, it is easy to see that these relations determine the correlators up to a finite subset. Strictly speaking, the instanton correlators $\langle \prod_{a=1}^u \sigma_a^{A_a} \rangle$ are only defined for $A_a \geq 0$ and $\sum_a A_a \geq d$, and if one attempts to extend the A_a outside the physical range by using the quantum cohomology relations, it is possible that one will run into inconsistencies. However, when the GLSM has, in addition to a pure Higgs phase, a phase with isolated

Coulomb vacua, we believe the relations can be extended consistently to generate a full $\mathbb{Z}^u$ lattice of functions which obey the quantum cohomology relations and the ghost number selection rule. We will call this property *extendability*. We will show below that this property holds in a wide class of examples.

Extendability is a necessary condition for the following conjecture: the A-model correlators of a toric GLSM with a pure Higgs phase and a phase with isolated Coulomb vacua can be put into the general form

$$Y_{A_1,\ldots,A_u} = \langle \prod_{a=1}^{u} \sigma_a^{A_a} \rangle = \sum_{\alpha=1}^{N_v} Z(\sigma_{1,\alpha},\ldots,\sigma_{u,\alpha}) \prod_a \sigma_{a,\alpha}^{A_a}, \tag{41}$$

where $(\sigma_{1,\alpha},\ldots,\sigma_{u,\alpha})$, $\alpha = 1,\ldots,N_v$ are the isolated Coulomb vacua.

It should be possible to prove this in all generality. However, as we understand it now, the proof is likely to require some more general tools of Gröbner bases and elimination theory. We will content ourselves with a proof of the conjecture for the case $u = 2$.

Consider a toric GLSM with $u = 2$ which satisfies the assumptions above and, in addition, has an irreducible $P(\omega)$. The presence of the pure Higgs phase allows us to assert that the A-model correlators are rational functions of q_1, q_2, satisfying eqn. (39) and obeying the selection rule that $Y_{A,B} = 0$ unless $A + B = d + \Delta n_1$. Since the GLSM also has a phase with isolated Coulomb vacua, it follows that a candidate set of correlators that obey the relations is given in eqn. (41), with the $N_v = \deg(P)|\Delta|$ vacua determined by the twisted equations of motion. If the GLSM correlators are extendable to $(A, B) \in \mathbb{Z}^2$, then it is possible that there exists a $Z(\sigma_1, \sigma_2)$ such that the correlators are computed by eqn. (41). We will now construct the required $Z(\sigma_1, \sigma_2)$.

The selection rule $A + B = d + \Delta n_1$ implies that

$$Z(\sigma_1, \sigma_2) = \frac{1}{|\Delta|}\sigma_1^{-d} F(\omega), \tag{42}$$

in which case eqn. (41) reduces to

$$Y_{d+\Delta n_1 - B,B} = \left(\mu^\Delta q_1\right)^{n_1} \sum_{\omega|P(\omega)=0} s(\omega)^{n_1} \omega^B F(\omega). \tag{43}$$

If we label the roots of $P(\omega)$ by ω_s, $s = 1,\ldots,\deg(P)$, then it is straightforward to determine $F_s = F(\omega_s)$ in terms of the $Y_{d-B,B}$: we merely need to solve the linear system

$$\sum_s M_{ts} F_s = Y_{d-t,t}, \tag{44}$$

where $M_{ts} = (\omega_s)^t$ is the Vandermonde matrix for the polynomial $P(\omega)$. M is invertible provided that the discriminant of P is non-zero. This will be true for generic q_2. Having determined the F_s, we can construct an $F(\omega)$:

$$F(\omega) = \sum_{s=1}^{\deg(P)} \frac{P(\omega) F_s}{(\omega - \omega_s) P'(\omega_s)}. \tag{45}$$

Obviously, $F(\omega)$ is *not* determined uniquely: the correlators are unchanged under the shift $F(\omega) \to F(\omega) + P(\omega)g(\omega)$, where $g(\omega)$ is any function that is non-singular at the roots of $P(\omega)$.

This elegant form depends on the assumption that $P(\omega)$ is irreducible. This assumption was made in a slightly more general form in writing a polynomial form for the equations of motion (eqn. (13)). It seems that the proper generalization is to work with $\hat{P}(\omega)$, the irreducible, q_2 dependent factor of $P(\omega)$, as the roots of $\hat{P}(\omega)$ will actually correspond to the σ-vacua. It is conceivable that in the case that $P(\omega)$ is reducible, one may be able to write down a *stronger* set of quantum cohomology relations than the ones in eqn. (39). However, these stronger relations are not needed for any of the arguments above. Thus, we expect that the reduction to $\hat{P}(\omega)$ is all that is necessary to handle the case of reducible $P(\omega)$.[8]

Although our approach uses the quantum cohomology relations to argue that A-model correlators are computed by the Coulomb branch in the IR phases, and we cannot explicitly perform the Coulomb branch computation, it is interesting to consider the converse argument: if the A-model correlators are computed by the Coulomb branch, then they obey the quantum cohomology relations. If true, it would provide a physical explanation for the quantum cohomology relations, which are very obscure from the point of view of the toric geometry of the $\mathcal{M}_{\mathbf{n}}$.

5 Two-parameter models

The reader may wonder to what extent the physics of the example of the $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$ GLSM resembles the general GLSM. In this section we will show that some generalizations are straightforward and enlightening. We stick to two-parameter models because of their simplicity.

5.1 General orbifold GLSMs with $u = 2$

The simplest generalization is to consider the most general two-parameter model with a $\mathbb{C}^3/\mathbb{Z}_{N(a,b,c)}$ phase. The toric fan of the minimal model for these GLSMs is easily written down. It has the same combinatorics as the toric fan of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$, except that the one dimensional cones are given by[9]

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} km-1 & 0 & 0 \\ 0 & km-1 & 0 \\ 0 & 0 & km-1 \\ k & k & 1 \\ 1 & 1 & m \end{pmatrix}, \tag{46}$$

[8] This is the description for $u = 2$. Its generalization to $u > 2$ is obvious.

[9] We do not prove this, but it can probably be proved by using the results of Sebö on Hilbert bases [21].

where k, m are positive integers and $km > 1$. The charges that follow from this fan are

$$Q = \begin{pmatrix} 0 & 0 & 1 & 1 & -k \\ 1 & 1 & 0 & -m & 1 \end{pmatrix}. \tag{47}$$

This explicit form of Q is useful. For example, the results of the previous section relied on $P(\omega)$ being an irreducible polynomial. Due to the simple form of $P(\omega)$ (eqn. (15)), it is easy to state this as a condition on the $\mathbf{Q}_i$: if $Q_i^2 \neq 0$, then there does not exist $\alpha \in \mathbb{Q}_{>0}$ such that $\mathbf{Q}_i = -\alpha \mathbf{Q}_j$ for some j. The existence of such an α is, of course, independent of the choice of basis for the $\mathbf{Q}_i$. Thus, from eqn. (47), it follows that $P(\omega)$ has no q_2-independent roots, and the results of section 4 apply to these models directly.

Applying the standard techniques outlined in section 3, we can see that the phase diagram is similar to the example we have studied. Even without explicitly studying the correlators in these models, it is clear that their general structure would follow that of the example. Nevertheless, studying this set of models in greater detail may be a way towards understanding the form of Z in eqn. (41) directly from the combinatorial data of the GLSM. This would enable us to write down the A-model correlators without ever doing an instanton sum. We will realize this hope, at least for a large class of GLSMs, in the next section.

5.2 Topological correlators in orbifold models

Suppose we are given a GLSM with a phase where the A-model correlators may be computed by instanton sums. If we were able to write down a general expression like eqn. (29) for a (large enough) subset of these correlators, then, since these satisfy the quantum cohomology relations, we could use the results of section 4 to extract the $Z(\sigma_1, \sigma_2)$ or, equivalently, $F(\omega)$. This is a formidable task, since for arbitrary GLSMs, even with $u = 2$, the intersection theory of $\mathcal{M}_\mathbf{n}$ can be quite complicated. However, there is one situation where the $\mathcal{M}_\mathbf{n}$ have a simple structure— when the Higgs phase $\mathcal{K}$ corresponds to an orbifold. In fact, a few more assumptions are needed to make the computation tractable, but with these in hand, we will be able to extract $F(\omega)$.

Suppose we are given a GLSM with charges Q_i^a, $i = 1, \ldots, 5$ (generalization to $i = 1, \ldots, n$ is straightforward), and $a = 1, 2$. Suppose that, in addition, the GLSM satisfies the following properties:

- The Q_i^a are in a standard basis, with $\Delta = \sum_i Q_i^1 < 0$;
- the phase diagram has a phase $\mathcal{K}$ as shown in Figure (2).

From these assumptions it immediately follows that for any non-zero $\mathbf{n} \in \mathcal{K}^\vee$, $d_i = \langle \mathbf{n}, \mathbf{Q}_i \rangle$ satisfy

$$d_i < 0 \text{ for } i \leq 3, \qquad d_i \geq 0 \text{ for } i = 4, 5. \tag{48}$$

With these assumptions, the non-zero correlators take the form

$$Y_{3+\Delta n_1 - b, b} = \left(\mu^\Delta q_1(\mu)\right)^{n_1} \sum_{n_2 = N_-}^{N_+} q_2^{n_2} Y^{n_1, n_2}_{3+\Delta n_1 - b, b}, \tag{49}$$

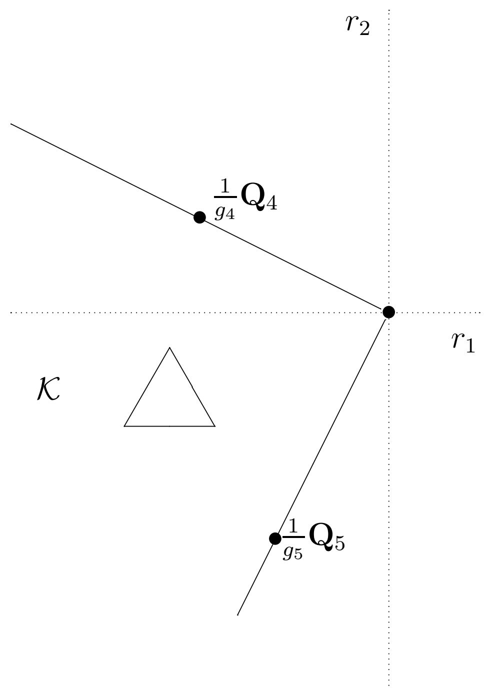


Figure 2: The orbifold phase in a general two-parameter model. $g_{4,5} = \gcd(Q^1_{4,5}, Q^2_{4,5})$.

where $N_+ = \lfloor -\frac{Q^1_5}{Q^2_5} n_1 \rfloor$, and $N_- = \lceil -\frac{Q^1_4}{Q^2_4} n_1 \rceil$.[10] The coefficients $Y^{n_1,n_2}_{3+\Delta n_1 - b, b}$ are determined by a tractable intersection computation on $\mathcal{M}_{\mathbf{n}} \simeq \mathbb{P}^{d_4} \times \mathbb{P}^{d_5}$. Omitting some straightforward steps (illustrated for the example of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$ in appendix B), we find that for $n_1 < 0$ correlators,

$$Y_{3+\Delta n_1 - b, b} = (\mu^\Delta q_1)^{n_1} \oint_{C(\omega_*)} \frac{d\omega}{2\pi i} \omega^b \prod_i \zeta_i^{-Q^1_i n_1 - 1} \prod_{Q^2_i > 0} \zeta_i^{Q^2_i} \left(q_2 \prod_i \zeta_i^{-Q^2_i} \right)^{N_-} P(\omega)^{-1}, \tag{50}$$

where $\omega_* = -Q^1_5/Q^2_5$. The integrand has, in addition to the pole at $\omega = -Q^1_5/Q^2_5$, a set of poles at the roots of $P(\omega)$ and a set of potential q_2-independent poles at

$$\begin{aligned} \omega &= \infty, \\ \zeta_i &= 0, \quad i \leq 4. \end{aligned}$$

We will now show that for $n_1 < 0$ these potential poles are not poles at all. First, for large ω, the integrand scales like $\omega^{A_\infty - 1} d\omega$, with

$$A_\infty = 1 + b - \deg(P) + \sum_{i | Q^2_i > 0} Q^2_i + \sum_{i | Q^2_i \neq 0} \left(-Q^1_i n_1 - 1 - Q^2_i N_- \right). \tag{51}$$

[10] Recall that $\lfloor a \rfloor$ ($\lceil a \rceil$) is the greatest (smallest) integer less (greater) than a.

Since $\deg(P) = \sum_{i|Q_i^2>0} Q_i^2$ and $0 \le b \le 3 + n_1\Delta$,

$$A_\infty \le -1 + \sum_{i|Q_i^2=0} (Q_i^1 n_1 + 1). \tag{52}$$

But,

$$\sum_{i|Q_i^2=0} (Q_i^1 n_1 + 1) = \sum_{i|Q_i^2=0, i<4} (\langle (n_1, 0), \mathbf{Q}_i \rangle + 1) \le 0, \tag{53}$$

and since $(n_1, 0) \in \mathcal{K}^\vee$, it follows from eqn. (48) that $A_\infty \le -1$, and there is no pole at $\omega = \infty$. Next, near $\zeta_i = 0$, $i \le 3$, the integrand scales as $\zeta_i^{A_i}$, with

$$A_i \ge -Q_i^1 n_1 - 1 - Q_i^2 N_- = -1 - \langle (n_1, N_-), \mathbf{Q}_i \rangle, \tag{54}$$

and since $(n_1, N_-) \in \mathcal{K}^\vee$, eqn. (48) guarantees $A_i > -1$. Finally, consider the integrand near $\zeta_4 = 0$. It scales as $\zeta_4^{A_4}$ with

$$A_4 = -Q_4^1 n_1 - 1 + Q_4^2 - Q_4^2 \lceil -\frac{Q_4^1}{Q_4^2} n_1 \rceil \ge Q_4^2 - 1 > -1, \tag{55}$$

since $Q_4^2 > 0$ by assumption. It follows that we may write the correlators as[11]

$$Y_{3+\Delta n_1 - b, b} = -\left(\mu^\Delta q_1\right)^{n_1} \sum_{\hat\omega | P(\hat\omega)=0} \oint_{C(\hat\omega)} \frac{d\omega}{2\pi i} s(\omega)^{n_1} \omega^b \frac{\prod_{i|Q_i^2>0} \zeta_i^{Q_i^2}}{P(\omega) \prod_i \zeta_i}, \tag{56}$$

where, as before, $s(\omega) = \prod_i \zeta_i^{-Q_i^1}$. Thus, we have an explicit demonstration that the $n_1 < 0$ correlators in these models satisfy the quantum cohomology relations, and the relations are extendable to all a, b that satisfy $a + b = 3 + \Delta n_1$. This means the correlators may be written in the form of eqn. (43). It is important to note that the $n_1 = 0$ correlators are not computed correctly by eqn. (50) with $n_1 = 0$, as, in particular, such expressions would not satisfy the quantum cohomology relations. However, they are correctly computed by eqn. (56). The difference between the two expression can be traced to the appearance of extra poles in the integrand of eqn. (50) when $n_1 = 0$.

We now wish to compare eqn. (56) to eqn. (43), which may be written as

$$Y_{3+\Delta n_1 - b, b} = \left(\mu^\Delta q_1\right)^{n_1} \sum_{\hat\omega | P(\hat\omega)=0} \oint_{C(\hat\omega)} \frac{d\omega}{2\pi i} s(\omega)^{n_1} \omega^b F(\omega) \frac{P'(\omega)}{P(\omega)}. \tag{57}$$

We may now extract $F(\omega)$:

$$F(\omega) = -\frac{\prod_{Q_i^2>0} \zeta_i^{Q_i^2}}{P'(\omega) \prod_i \zeta_i}. \tag{58}$$

[11] We have left off the factor of $\left(q_2 \prod_i \zeta_i^{-Q_i^2}\right)^{N_-}$, since at $\omega = \hat\omega$ it is 1.

It can be shown that when $P(\omega) = 0$,

$$P'(\omega) = \prod_{i|Q_i^2>0} \zeta_i^{Q_i^2} \sum_i \left(Q_i^2\right)^2 \zeta_i^{-1}, \tag{59}$$

so, finally,[12]

$$F(\omega) = \frac{\mathrm{sign}(\Delta)}{\sum_i \left(Q_i^2\right)^2 \prod_{j\neq i} \zeta_j}, \tag{60}$$

and the corresponding $Z(\sigma_1, \sigma_2)$ is

$$Z(\sigma_1, \sigma_2)^{-1} = \Delta \sigma_1^3 \sum_i (Q_i^2)^2 \prod_{j\neq i} \zeta_j \Bigg|_{\omega=\sigma_2/\sigma_1}. \tag{61}$$

5.3 A few examples

The example of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$ satisfies all the assumptions for our general treatment, and it is easy to verify that eqn. (61), specialized to this case is eqn. (38). What is more surprising is that eqns. (60,61) give the correct correlators in more general models. We have studied the following cases in detail.

- The $\mathbb{C}^3/\mathbb{Z}_{3(211)}$ GLSM. This model has charges

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & -2 & 1 \end{pmatrix}. \tag{62}$$

  Thus, $\Delta = 1 > 0$, and we compute the instanton sum in the smooth phase, where there exist sub-cones in $\mathcal{K}^\vee$—the d_i change signs in $\mathcal{K}^\vee$. The explicit computation of the instantons yields

$$F(\omega) = -\frac{1}{2\,(1+\omega)}. \tag{63}$$

- The $\mathbb{C}^3/\mathbb{Z}_{5(211)}$ GLSM. This model has charges

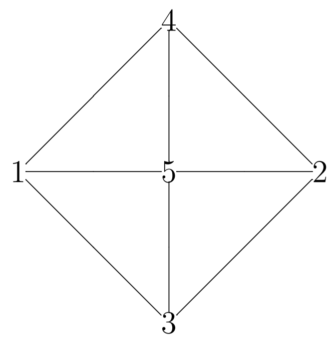
$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -2 & -1 \end{pmatrix}. \tag{64}$$

  So, $\Delta = 1 > 0$ again. The model has a unique UV phase—one of the partially resolved phases, and the instanton sum in this phase also has sub-cones. The smooth and orbifold phases are intermediate (in the sense of the RG flow). When we perform the instanton sums, we find

$$F(\omega) = -\frac{1}{2\,(1+5\omega+\omega^2)\,(1+\omega)}. \tag{65}$$

[12]The capricious way of writing -1 as $\mathrm{sign}(\Delta)$ will become apparent in the next section.

- Finally, consider a GLSM without a $\mathbb{C}^3/\mathbb{Z}_N$ orbifold phase. This model has the fan



and charges

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & -N \\ 1 & 1 & -1 & -1 & 0 \end{pmatrix}. \tag{66}$$

For $N > 2$, the model has a unique IR phase (the smooth phase) and two UV phases. Performing the instanton sums in one of the UV phases, we find

$$F(\omega) = -\frac{1}{2N\omega\,(1+\omega)}. \tag{67}$$

The reader can easily verify that in these three seemingly different cases, the $F(\omega)$ computed from the instanton expansion is precisely the $F(\omega)$ computed by eqn. (60).

These results lead us to a conjecture: Given a two-parameter GLSM with a standard basis of charges, $\Delta \neq 0$, and an irreducible (over $\mathbb{Z}$) $P(\omega)$, the non-zero A-model correlators are given by

$$Y_{a,b} = \sum_{\hat{\omega}|P(\hat{\omega})} s(\hat{\omega})^{n_1}\hat{\omega}^b F(\hat{\omega}), \quad \text{for } \ a+b = d + \Delta n_1, \tag{68}$$

where

$$\begin{aligned} s(\omega) &= \mu^{\Delta} q_1(\mu) \prod_i \zeta_i^{-Q_i^1} \\ F(\omega) &= \frac{\text{sign}(\Delta)}{\sum_i \left(Q_i^2\right)^2 \prod_{j\neq i} \zeta_j}, \end{aligned} \tag{69}$$

and $\zeta_i = Q_i^1 + Q_i^2\omega$.

The GLSM is known to be democratic in certain aspects—for example, a Calabi-Yau model seems to contain very different phases: some may have a nice geometric interpretation, others may be Landau-Ginzburg theories, and yet others may be exotic mixtures of the two. Despite this diversity before our undiscerning eyes, the GLSM shows us that certain aspects, like the chiral ring, remain independent of the particular phase. The seemingly universal form of $F(\omega)$ is another manifestation of the GLSM's democratic principles.

6 Discussion

We have spent the bulk of the paper exploring the properties of A-model observables in two-parameter GLSMs. We have found that these observables are supported by the Higgs branch in the UV, and we have shown that, while they cannot be supported by the Higgs branch in IR, it is plausible that they are supported by the isolated Coulomb vacua. In this section, we would like to discuss this and related findings from several perspectives, including a possible space-time interpretation for our two-dimensional findings.

6.1 Instanton sums in "wrong" phases

We have seen that in models with $\Delta \neq 0$ the ghost number selection rule (eqn. (24)) ensures that, while the UV phase Higgs branch supports an infinite number of correlators, the IR Higgs branch cannot support most of these. Of course, as the astute reader has noted, the $n_1 = 0$ correlators are not excluded by this rule, and we could attempt to calculate them in other phases. Let us consider the case of the smooth phase in the example of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$ studied in section 3. It is easy to see that here all instantons with $n_2 \geq 0$ should contribute. The standard techniques of Morrison and Plesser determine the $n_2 > 0$ contributions, while the $n_2 = 0$ contributions are fixed by assuming that the correlators satisfy the quantum cohomology relations of eqn. (31). The final result is that the computations of the $n_1 = 0$ instanton sums in the smooth phase differ by a sign from the $n_1 = 0$ instanton sums in the IR phase. In a sense, a difference of this sort should not surprise us: we know that there are Coulomb vacua in this phase, and the instanton sum cannot be reliable. In fact, it is more surprising that the difference is one of merely a minus sign. In many other cases, like that of the $\mathbb{C}^3/\mathbb{Z}_{3(211)}$ model mentioned above, the difference is more drastic. There, an instanton computation of the $Y_{3-b,b}$ correlators in the IR phase would only have contributions from $\mathbf{n} = 0$ instantons, yet the UV computation results in $Y_{3-b,b}$ with a non-trivial q_2 dependence.

This suggests that all of the topological correlators, including those that are "allowed" to be calculated on the IR Higgs branch are, in fact, supported on the IR Coulomb branch.

In addition, one might be tempted to try to compute instanton sums in intermediate phases. We have not studied this in detail, but some preliminary investigations have suggested that here the instanton sums lead to expressions that are non-rational functions of the q_a. It seems that this can again be attributed to the existence of Coulomb vacua in these phases.

6.2 Singularities of Coulomb Branch Correlators

Although we have shown that an instanton computation is inappropriate in IR phases, and we have attributed this to the emergence of the Coulomb branch, the reader has surely noted that we have not performed a calculation on the Coulomb branch to verify our findings. Instead, we argued that the form of the correlators that follows from the quantum cohomology relations is consistent with the expected form of a Coulomb branch computation.

The Higgs computation of the correlators had a very nice feature: the singularities in the correlators were associated to a reasonable physical phenomenon: the un-Higgsing of a gauge group and the emergence of the continuous Coulomb branch. Can we make a similar association for the isolated Coulomb computation of the correlators? This is particularly urgent if, as in the previous section, we argue that even the $n_1 = 0$ correlators cannot be computed by an instanton sum in the IR.

Restricting attention to the $u = 2$ case, it is clear that singularities in the Higgs correlators are due to the un-Higgsing of the gauge subgroup with charges Q_i^2, while keeping σ_1 massive. This will occur when at least one matter multiplet is uncharged under this subgroup. In that case, the semi-classical prediction of this singularity places it at

$$q_2^* = \prod_{i|Q_i^2 \neq 0} \left(Q_i^2\right)^{Q_i^2}. \tag{70}$$

As q_2 approaches q_2^*, then some of the isolated Coulomb vacua have the property that ω is sent to ∞, and, since $s(\omega) \sim \omega^{-\Delta}$ for large ω, we see that $\sigma_1 \sim \omega^{-1}$ and $\sigma_2 \sim$ const.

Recall, that the continuous Coulomb branch has $\sigma_1 = 0$, and it can be reliably studied by the effective twisted superpotential when σ_2 is large. If the continuous Coulomb branch continues to finite σ_2, then is seems sensible to attribute the singularity in the correlators to some of the isolated vacua approaching the continuous Coulomb branch.

The isolated Coulomb vacua can obviously have a different type of seemingly pathological behavior: as q_2 is varied, some of the isolated vacua will merge. This will happen when $\operatorname{discrim}_\omega P(\omega) = 0$. Since $F(\omega) \sim P'(\omega)^{-1}$, it is clear that near these q_2 the weighting factor $Z(\sigma_1, \sigma_2)$ will diverge, yet these divergences do not show up in the A-model correlators.[13]It would be interesting to understand why the merging of isolated and continuous Coulomb branches seems to lead to singularities in correlators, while the merging of two isolated Coulomb vacua does not lead to singularities in the correlators.

6.3 General A-model correlators and $F(\omega)$

We have made some progress towards computing A-model correlators in general toric GLSMs. Our analysis of GLSMs with a general orbifold phase covers all two-parameter models with applications to localized tachyon condensation. Furthermore, if our conjecture for the form of $F(\omega)$ is correct, then we have determined the A-model correlators in all two-parameter models.

It would also be very useful to generalize the analysis beyond $u = 2$. Two parameter models are quite special: for example, once the standard basis is chosen for Q_i^a, Q_i^2 are essentially unique. Even more fundamentally, all cones in $\mathbb{R}^2$ are simplicial, which greatly simplifies the structure of the instanton sums. While much of our approach is tractable precisely because $u = 2$, it seems possible that more general techniques like Gröbner bases and elimination theory may still be tractable due to the underlying toric structure of the problem [22].

[13]To be precise, this is true for $q_2 \neq 0, \infty$.

6.4 Towards a Space-time Interpretation

Although properties of RG flows in $d = 2$ theories are certainly interesting, the real interest of this project lies in the space-time interpretation of the world-sheet physics. Unfortunately, this is difficult for several reasons. First, although the chiral ring of the GLSM is readily identified with the chiral ring of the UV orbifold theory, and the quantum cohomology relations of the GLSM correspond to the deformed chiral ring relations of the orbifold, due to the absence of space-time supersymmetry, it remains difficult to understand what precisely the chiral ring is computing in space-time. The second, and perhaps more serious, problem stems from the fact that, unlike in the case of open string tachyons, the IR fixed point seen in the GLSM RG flow is certainly *not* the endpoint of tachyon condensation [4]. At best, it should be thought of as a description suitable for many decades of the RG time. Thus, we cannot match the chiral ring that we track through the flow to some sensible static space-time physics in the IR. Instead, we must try to interpret our results in some non-static background. This seems perilous indeed, but let us attempt it.

The interpretation that the GLSM suggests is that the IR phase corresponds to an expanding space-time, where divisors are getting large, and any non-trivial α' effects are suppressed. If we trust this picture, it seems natural to suggest that the Coulomb vacua and the chiral ring they seem to support should be associated to the "edge" of this expanding space-time. This would easily explain why these effects are difficult to interpret from the point of view of the IR Higgs branch, which, one would think, should be associated to the physics discernible to an observer inside the expanding bubble. This interpretation is consistent with the decoupling of the two IR branches that could be seen from a naive classical argument that neglected Kähler term renormalization.

If this interpretation is reasonable, then it is clear that the difficulties of interpreting the Coulomb branch physics are tied to the non-compactness of the models we have been considering. It would be interesting to construct models where our framework describes some local geometry in a compact space. One could hope that with a suitable set of limits, one would be able to describe the physics of the Coulomb branch in terms of the compact space.

While such efforts are worthwhile, and may at least reveal whether the proposed connection is on the right track, they will still be within the limitations of the RG approach to tachyon condensation: while providing a heuristically reasonable picture, there are great difficulties in relating the RG flow to space-time physics. As we have shown, the RG flow of the GLSM contains a lot of rich physics, and topological methods can be used to probe these in a quantitative fashion. New methods are needed to go beyond heuristics in translating these world-sheet riches into space-time gains.

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A Some Toric Geometry

To make our work more self-contained, we provide some additional details about the structure of the instanton moduli spaces $\mathcal{M}_{\mathbf{n}}$.

Let $d_i = \langle \mathbf{n}, \mathbf{Q}_i \rangle$ and $I_{\mathbf{n}} = \{i | d_i < 0\}$. Elaborating on eqn. (25), $\mathcal{M}_{\mathbf{n}}$ is a toric variety of dimension $\dim(\mathcal{M}_{\mathbf{n}}) = \sum_{i \notin I_{\mathbf{n}}} d_i - u$, defined by the homogeneous coordinates ξ_{ij}, $i \notin I_{\mathbf{n}}$, $j = 0, \ldots, d_i$ with charges Q_i^a under $(\mathbb{C}^*)^u$ and an excluded set $F_{\mathbf{n}}$. The intersection ring of $\mathcal{M}_{\mathbf{n}}$ is generated by classes $\sigma_a^{\mathbf{n}}$, in terms of which the divisor $\xi_{ij} = 0$ is

$$\xi_i^{(\mathbf{n})} = \sum_a Q_i^a \sigma_a^{(\mathbf{n})}. \tag{71}$$

We will use the notation $\xi_i^{(\mathbf{n})}$ for this linear combination of generators of the intersection ring even for $i \in I_{\mathbf{n}}$. Note that (up to scaling by μ) these divisors are obtained when, following section 2.6, ξ_i is mapped as a polynomial in σ_a to a divisor in $\mathcal{M}_{\mathbf{n}}$.

$F_{\mathbf{n}}$ may be complicated, but a certain subset of it is determined by the cone $\mathcal{K}$. $\mathcal{K}$ may be presented as the set of points in $\mathbb{R}^u$ satisfying a set of inequalities:

$$\mathcal{K} = \{\mathbf{r} \in \mathbb{R}^u | \langle \mathbf{m}^\alpha, \mathbf{r} \rangle > 0, \alpha = 1 \ldots A\}. \tag{72}$$

Then

$$\left\{\xi_{i\ 1} = \xi_{i\ (d_i+1)} = 0 | \langle \mathbf{m}^\alpha, Q_i \rangle > 0 \ \text{ and } \ i \in I_{\mathbf{n}}\right\} \subset F_{\mathbf{n}}. \tag{73}$$

As explained in [15], each of these A intersections of hyperplanes leads to a (Stanley-Reisner) relation on the intersection ring of $\mathcal{M}_{\mathbf{n}}$:

$$\prod_{\substack{d_i \geq 0 \\ \langle \mathbf{m}^\alpha, Q_i \rangle > 0}} \left(\xi_i^{(\mathbf{n})}\right)^{d_i+1} = 0. \tag{74}$$

When little or no confusion can arise, we will drop the $\mathbf{n}$ superscripts on the divisors $\xi_i^{\mathbf{n}}$.

Modulo the nontrivial scaling by μ discussed in section 2.6, the class $\mathcal{O}^{(\mathbf{n})}$ is obtained from $\mathcal{O}$ expressed as a polynomial in σ_a by writing the same polynomial in $\sigma_a^{(\mathbf{n})}$.

The Euler class is constructed from the divisors $\xi_i^{\mathbf{n}}$ with $d_i < 0$:

$$\chi_{\mathbf{n}} = \prod_{i \in I_{\mathbf{n}}} \left(\xi_i^{(\mathbf{n})}\right)^{-1-d_i}. \tag{75}$$

Its degree is $\sum_{i \in I_{\mathbf{n}}} (-1 - d_i)$. One can check that this is consistent with the ghost number selection rule: unless

$$\deg(\mathscr{O}) + \sum_{i \in I_{\mathbf{n}}} (-1 - d_i) = \dim(\mathcal{M}_{\mathbf{n}}) = \sum_{i \notin I} (d_i + 1) - u, \tag{76}$$

instantons with instanton number $\mathbf{n}$ do not contribute to $Y_{\mathscr{O}}$.

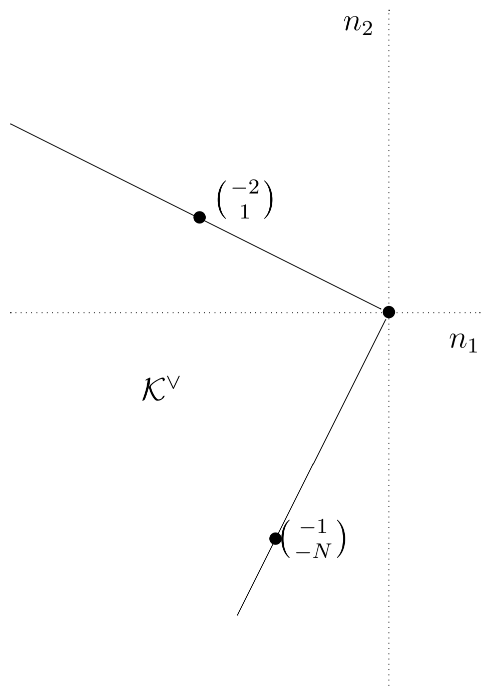

Figure 3: The dual cone in the orbifold phase of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$.

B An instanton computation

The reader may be interested in how we obtain eqns. (30, 50). In this appendix we will fill in some of the steps of this computation. We will work with the example of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$. First, we will show how we may derive a formula for the instanton contribution $Y^{\mathbf{n}}_{a,b}$, and then we will show how the instanton sum can be manipulated into the form of eqn. (50). We will use the results and notation of appendix A.

To sum the instantons in the orbifold phase of $\mathbb{C}^3/\mathbb{Z}_{(2N+1)(2,2,1)}$, we begin by determining the dual cone $\mathcal{K}^\vee$ of contributing instantons n. $\mathcal{K}$ determines $\mathcal{K}^\vee$ to be the cone shown in figure 3. Thus, the contributing instantons have $n_1 < 0$ and $Nn_1 \leq n_2 \leq \lfloor -\frac{n_1}{2} \rfloor$. Since

$$d_i = (n_1, n_1, n_1 + n_2, n_2 - Nn_1, -n_1 - 2n_2)\,, \tag{77}$$

we see that

$$F_{\mathbf{n}} = \{\xi_{4j} = 0 | j = 0, \ldots, d_4\} \cup \{\xi_{5j} = 0 | j = 0, \ldots, d_5\}\,. \tag{78}$$

This leads to the Stanley-Reisner relations on the intersection ring of $\mathcal{M}_{\mathbf{n}}$:

$$\begin{aligned} \left(\sigma_2^{(\mathbf{n})} - N\sigma_1^{(\mathbf{n})}\right)^{d_4+1} &= 0, \\ \left(-\sigma_1^{(\mathbf{n})} - 2\sigma_2^{(\mathbf{n})}\right)^{d_5+1} &= 0. \end{aligned} \tag{79}$$

We may also write down the Euler class:

$$\chi_{\mathbf{n}} = \prod_{i\leq 3} \left(\xi_i^{(\mathbf{n})}\right)^{-d_i-1}. \tag{80}$$

We will henceforth drop the superscript $(\mathbf{n})$ on the various divisors. The σ_1, σ_2 basis for the divisors is a bit inconvenient for the purposes of writing down the $Y^{\mathbf{n}}_{a,b}$. A more convenient basis is

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \xi_4 \\ \xi_5 \end{pmatrix} = \begin{pmatrix} -N & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}. \tag{81}$$

In this basis the Stanley-Reisner relations are $\lambda_1^{d_4+1} = 0$ and $\lambda_2^{d_5+1} = 0$, and the moduli space is $\mathcal{M}_{\mathbf{n}} = \mathbb{P}^{d_4} \times \mathbb{P}^{d_5}$, with the λ_1, λ_2 being the fundamental classes on the $\mathbb{P}$s. Writing the Euler class and the σ_1, σ_2 in terms of the λ_1, λ_2, eqn. (26) becomes

$$\begin{aligned} Y^{\mathbf{n}}_{a,b} &= \frac{1}{M}\langle \sigma_1^a \sigma_2^b \chi_{\mathbf{n}} \rangle_{\mathcal{M}_{\mathbf{n}}} \\ &= \frac{1}{M}\langle \left(\frac{2\lambda_1+\lambda_2}{-M}\right)^{a-2n_1-2} \left(\frac{-\lambda_1+N\lambda_2}{-M}\right)^{b} \left(\frac{\lambda_1+(N+1)\lambda_2}{-M}\right)^{-n_1-n_2-1} \rangle_{\mathcal{M}_{\mathbf{n}}}. \end{aligned} \tag{82}$$

We have let $M = 2N+1$, and the leading factor of $\frac{1}{M}$ is due to over-counting the instantons because of the remaining discrete gauge symmetry in the orbifold phase. The intersection theory on $\mathcal{M}_{\mathbf{n}}$ is extremely simple, and $Y^{\mathbf{n}}_{a,b}$ is just the coefficient of $\lambda_1^{d_4}$ in the above product of binomials. Note that if $a+b = 3+(2-N)n_1$, then the power of λ_2 is correct, and otherwise $Y^{\mathbf{n}}_{a,b}$ vanishes. By letting $\lambda_1 = 1$ and $\lambda_2 = z$, we can write the coefficient of $\lambda_1^{d_4}\lambda_2^{d_5}$ as a contour integral about $z = 0$. Trivial manipulations then yield eqn. (30), which we reproduce here for convenience:

$$\begin{aligned} Y^{n_1,n_2}_{a,b} &= \mu^{n_1(2-N)}\tfrac{1}{M}\oint_{C(0)} \frac{dz}{2\pi i}\frac{(2+z)}{z(1+(N+1)z)}\left[\frac{(-M)^{N+1}z}{(2+z)^N(1+(N+1)z)}\right]^{n_1} \times \\ &\times \left[\frac{-Mz^2}{1+(N+1)z}\right]^{n_2}\left[\frac{-1+Nz}{2+z}\right]^{b}. \end{aligned} \tag{83}$$

We will now work further with this explicit form to show in an example how to obtain eqn. (50). We begin by performing the sum on n_2, which runs for $Nn_1 \leq n_2 \leq \lfloor -\frac{n_1}{2} \rfloor$:

$$\begin{aligned} Y_{3+(2-N)n_1-b,b} &= \left(\mu^{(2-N)}q_1\right)^{n_1}\tfrac{1}{M}\oint_{C(0)} \frac{dz}{2\pi i}\frac{(2+z)}{z(1+(N+1)z)}\left[\frac{(-M)^{N+1}z}{(2+z)^N(1+(N+1)z)}\right]^{n_1} \times \\ &\times \left[\frac{-1+Nz}{2+z}\right]^{b}\frac{R^{Nn_1}-R^{\lfloor -\frac{n_1}{2}\rfloor+1}}{1-R}. \end{aligned} \tag{84}$$

where $R = -Mz^2q_2/(1+(N+1)z)$. The term with $R^{\lfloor -\frac{n_1}{2}\rfloor+1}$ does not contribute to the residue at zero, so we may discard it. Now, we change coordinates to

$$w = \frac{-1+Nz}{2+z}. \tag{85}$$

This leads to a form familiar from eqn. (50):

$$Y_{3+(2-N)n_1-b,b} = \left(\mu^{2-N} q_1\right)^{n_1} \oint_{C\left(-\frac{1}{2}\right)} \frac{dw}{2\pi i} (-1-2w)^{(2N+1)n_1-1} (1+w)^{-(N+1)n_1} q_2^{Nn_1} \frac{w^b}{P(\omega)}. \quad (86)$$

It is evident that the poles of the integrand are $\omega = -1/2$ and the roots of $P(\omega)$, so, finally, we find

$$Y_{3+(2-N)n_1-b,b} = \mu^{N-2} q_1(\mu)^{-1} \sum_{\hat{\omega}|P(\hat{\omega})} \oint_{C(\hat{\omega})} \frac{dw}{2\pi i} \left(\frac{(-1-2\omega)(w-N)^N}{1+\omega} \right)^{n_1} \frac{w^b}{(1+2\omega)P(\omega)}. \quad (87)$$

Comparing to the standard form of the correlators, $F(\omega) = ((1+2\omega)P'(\omega))^{-1}$, which at the roots of $P(\omega)$ reduces to the familiar $F(\omega) = (1+3N+2N\omega)^{-1}$.

C A Proof of Quantum Cohomology Relations

In this appendix we prove that eqn. (39) holds, provided that the A-model correlators may be computed by the standard instanton sums in *some* phase $\mathcal{K}$. We will use notation given in the text as well as appendix A.

Expanding eqn. (39) in powers of q_a, and using eqn. (26), we find that eqn. (39) is equivalent to three statements. The first of these,

$$\langle \mathscr{O} \prod_{Q_i^a>0} \xi_i^{Q_i^a} \prod_{d_i<0} \xi_i^{-d_i-1} \rangle_{\mathcal{M}_{\mathbf{n}}} = \langle \mathscr{O} \prod_{Q_i^a<0} \xi_i^{-Q_i^a} \prod_{d_i-Q_i^a<0} \xi_i^{Q_i^a-d_i-1} \rangle_{\mathcal{M}_{\mathbf{n}-\mathbf{e}_a}}, \quad (88)$$

must hold when $\mathbf{n}$ and $\mathbf{n}-\mathbf{e}_a$ are both in $\mathcal{K}^\vee$. The second and third should hold when one of the vectors is in $\mathcal{K}^\vee$ while the other is not:

$$\langle \mathscr{O} \prod_{Q_i^a>0} \xi_i^{Q_i^a} \prod_{d_i<0} \xi_i^{-d_i-1} \rangle_{\mathcal{M}_{\mathbf{n}}} = 0, \quad \mathbf{n} \in \mathcal{K}^\vee, \mathbf{n}-\mathbf{e}_a \notin \mathcal{K}^\vee, \quad (89)$$

$$\langle \mathscr{O} \prod_{Q_i^a<0} \xi_i^{-Q_i^a} \prod_{d_i-Q_i^a<0} \xi_i^{Q_i^a-d_i-1} \rangle_{\mathcal{M}_{\mathbf{n}-\mathbf{e}_a}} = 0, \quad \mathbf{n} \in \mathcal{K}^\vee, \mathbf{n}-\mathbf{e}_a \notin \mathcal{K}^\vee. \quad (90)$$

To prove the first of these conditions, we will now establish some additional properties of the intersection ring on $\mathcal{M}_{\mathbf{n}}$. Suppose $\mathbf{n}, \mathbf{n}-\mathbf{e}_a \in \mathcal{K}^\vee$. Furthermore, consider the case that $I_{\mathbf{n}} = I_{\mathbf{n}-\mathbf{e}_a}$. Then for any $\mathcal{U}$,

$$\langle \mathcal{U} \prod_{\substack{Q_i^a>0 \\ d_i\geq 0}} \xi_i^{Q_i^a} \rangle_{\mathcal{M}_{\mathbf{n}}} = \langle \mathcal{U} \prod_{\substack{Q_i^a<0 \\ d_i\geq 0}} \xi_i^{-Q_i^a} \rangle_{\mathcal{M}_{\mathbf{n}-\mathbf{e}_a}}. \quad (91)$$

This follows because $d_i(\mathbf{n}-\mathbf{e}_a) = d_i - Q_i^a$ and the intersection of Q_i^a copies of ξ_i in $\mathcal{M}_{\mathbf{n}}$ effectively leaves a variety described by $d_i - Q_i^a$ copies of ξ_i—$\mathcal{M}_{\mathbf{n}-\mathbf{e}_a}$.

More generally, $I_{\mathbf{n}} \neq I_{\mathbf{n}-\mathbf{e}_a}$, and the above is generalized to

$$\langle \mathcal{U} \prod_{\substack{Q_i^a>0 \\ d_i \geq 0}} \xi_i^{\min(Q_i^a, d_i+1)} \rangle_{\mathcal{M}_{\mathbf{n}}} = \langle \mathcal{U} \prod_{\substack{Q_i^a<0 \\ d_i - Q_i^a \geq 0}} \xi_i^{\min(-Q_i^a, d_i+1-Q_i^a)} \rangle_{\mathcal{M}_{\mathbf{n}-\mathbf{e}_a}}. \tag{92}$$

To see this, observe that if $Q_i^a > d_i \geq 0$, then $\mathcal{M}_{\mathbf{n}-\mathbf{e}_a}$ contains *no* copies of ξ_i, which is achieved by restricting to the intersection of $d_i + 1$ copies of ξ_i in $\mathcal{M}_{\mathbf{n}}$.

A few moments' thought will convince the reader that

$$\prod_{Q_i^a > d_i \geq 0} \xi_i^{Q_i^a - d_i - 1} \prod_{\substack{Q_i>0 \\ d_i<0}} \xi_i^{Q_i^a} \prod_{d_i<0} \xi_i^{-d_i-1} = \prod_{d - Q_i^a < 0} \xi_i^{Q_i^a - d_i - 1} \prod_{\substack{Q_i^a<0 \\ d_i - Q_i^a < 0}} \xi_i^{-Q_i^a} \prod_{\substack{d_i - Q_i^a \geq 0 \\ d_i<0}} \xi_i^{-d_i-1}. \tag{93}$$

We are finally in a position to prove our claim. Consider

$$\mathcal{U} = \mathscr{O} \prod_{Q_i^a > d_i \geq 0} \xi_i^{Q_i^a - d_i - 1} \prod_{\substack{Q_i^a>0 \\ d_i<0}} \xi_i^{Q_i^a} \prod_{d_i<0} \xi_i^{-d_i-1}. \tag{94}$$

On one hand,

$$\mathcal{U} \prod_{\substack{Q_i^a>0 \\ d_i \geq 0}} \xi_i^{\min(Q_i^a, d_i+1)} = \mathscr{O} \prod_{Q_i^a>0} \xi_i^{Q_i^a} \prod_{d_i<0} \xi_i^{-d_i-1}. \tag{95}$$

On the other hand, using eqn. (93) in $\mathcal{U}$, we have

$$\mathcal{U} \prod_{\substack{Q_i^a<0 \\ d_i - Q_i^a \geq 0}} \xi_i^{\min(-Q_i^a, d_i+1-Q_i^a)} = \mathscr{O} \prod_{Q_i^a<0} \xi_i^{-Q_i^a} \prod_{d_i - Q_i^a<0} \xi_i^{Q_i^a - d_i - 1}. \tag{96}$$

Using eqn. (92), we see that eqn. (88) holds.

The second and third conditions are easier. Construct V, the toric variety associated to $\mathcal{K}$ as a Kähler quotient. The form of the D-terms (the moment map in geometric terminology) shows that the cone $\mathcal{K}$ is generated by a subset of the $\mathbf{Q}_i$, in fact, precisely those $\mathbf{Q}_i$ that satisfy $\langle \mathbf{m}^\alpha, \mathbf{Q}_i \rangle > 0$ for $\alpha = 1, \ldots, A$.[14] Thus, if $\mathbf{n} \in \mathcal{K}^\vee$, then for any i such that $\langle \mathbf{m}^\alpha, \mathbf{Q}_i \rangle > 0$ for all α, $d_i = \langle n, \mathbf{Q}_i \rangle \geq 0$. If, in addition, $\mathbf{n} - \mathbf{e}_a \notin \mathcal{K}^\vee$, then there exists some i such that $\langle \mathbf{m}^\alpha, \mathbf{Q}_i \rangle > 0$ and $d_i - Q_i^a = \langle \mathbf{n} - \mathbf{e}_a, Q_i \rangle < 0$. So, we conclude that if $\mathbf{n} \in \mathcal{K}^\vee$ and $\mathbf{n} - \mathbf{e}_a \notin \mathcal{K}^\vee$, then there exists some i such that $\langle \mathbf{m}^\alpha, \mathbf{Q}_i \rangle > 0$, and $Q_i^a > d_i \geq 0$. The second condition then follows as a consequence of the corresponding Stanley-Reisner relation in eqn. (74). The third condition is shown to hold in the same manner.

References

[1] A. Adams, J. Polchinski, and E. Silverstein. Don't panic! Closed string tachyons in ALE space-times. *JHEP*, 10:029, 2001, arXiv:hep-th/0108075.

[14] Recall that $\mathbf{m}^\alpha$ determined a set of inequalities that defined $\mathcal{K}$.

[2] C. Vafa. Mirror symmetry and closed string tachyon condensation. 2001, arXiv:hep-th/0111051.

[3] J.A. Harvey, D. Kutasov, E.J. Martinec, and G.W. Moore. Localized tachyons and RG flows. 2001, arXiv:hep-th/0111154.

[4] D.R. Morrison, K. Narayan, and M.R. Plesser. Localized tachyons in $\mathbb{C}^3/\mathbb{Z}_n$. *JHEP*, 08:047, 2004, arXiv:hep-th/0406039.

[5] T. Sarkar. On localized tachyon condensation in $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^3/\mathbb{Z}_n$. *Nucl. Phys.*, B700:490–520, 2004, arXiv:hep-th/0407070.

[6] D.R. Morrison and K. Narayan. On tachyons, gauged linear sigma models, and flip transitions. 2004, arXiv:hep-th/0412337.

[7] M. Headrick, S. Minwalla, and T. Takayanagi. Closed string tachyon condensation: An overview. *Class. Quant. Grav.*, 21:S1539–S1565, 2004", arXiv:hep-th/0405064.

[8] E.J. Martinec and G.W. Moore. On decay of K-theory. 2002, arXiv:hep-th/0212059.

[9] G. Moore and A. Parnachev. Localized tachyons and the quantum McKay correspondence. *JHEP*, 11:086, 2004, arXiv:hep-th/0403016.

[10] A. Sen. Tachyon dynamics in open string theory. 2004,arXiv:hep-th/0410103.

[11] L. Dixon, D. Friedan, E. Martinec, and S. Shenker. The conformal field theory of orbifolds. *Nucl. Phys.*, B282:13, 1987.

[12] A. Dabholkar, A. Iqubal, and J. Raeymaekers. Off-shell interactions for closed string tachyons. *JHEP*, 05:051, 2004, arXiv:hep-th/0403238.

[13] S. Lee and S.-J. Sin. Chiral rings and GSO projection in orbifolds. *Phys.Rev.*, D69:026003, 2003, arXiv:hep-th/0308029.

[14] E. Witten. Phases of $N = 2$ theories in two dimensions. *Nucl. Phys.*, B403:159–222, 1993, arXiv:hep-th/9301042.

[15] D.R. Morrison and M.R. Plesser. Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties. *Nucl. Phys.*, B440:279–354, 1995, arXiv:hep-th/9412236.

[16] J. Wess and J. Bagger. *Supersymmetry and Supergravity*. Princeton University Press, Princeton, New Jersey, 2nd edition, 1992.

[17] K. Hori and C. Vafa. Mirror symmetry. 2000, arXiv:hep-th/0002222.

[18] E. Witten. Mirror manifolds and topological field theory. 1991, arXiv:hep-th/9112056.

[19] P.S. Aspinwall and D.R. Morrison. Topological field theory and rational curves. *Commun. Math. Phys.*, 151:245–262, 1993, arXiv:hep-th/9110048.

[20] A. Craw. *The McKay correspondence and representations of the McKay quiver*. PhD thesis, University of Warwick, 2001.

[21] András Sebö. Hilbert bases, Caratheodory's theorem, and combinatorial optimization . In R. Kannan and W.R. Pulleyblank, editors, *Integer Programming and Combinatorial Optimization*, pages 431–455. University of Waterloo Press, 1990.

[22] B. Sturmfels. *Solving Systems of Polynomial Equations*. American Mathematical Society, Providence, Rhode Island, 2002.